

9.10 Double Integral

Functions of two variables: $f(x, y)$, $f(u, v)$, ...

Double integrals: $\iint_R f(x, y) dx dy$, $\iint_R g(x, y) dx dy$, ...

Riemann sum: $\sum_{i=1}^m \sum_{j=1}^n f(u_i, v_j) \Delta x_i \Delta y_j$

Small changes: Δx_i , Δy_j

Regions of integration: R , S

Polar coordinates: r , θ

Area: A

Surface area: S

Volume of a solid: V

Mass of a lamina: m

Density: $\rho(x, y)$

First moments: M_x , M_y

Moments of inertia: I_x , I_y , I_0

Charge of a plate: Q

Charge density: $\sigma(x, y)$

Coordinates of center of mass: $\bar{x}$, $\bar{y}$

Average of a function: μ

1078. Definition of Double Integral

The double integral over a rectangle $[a, b] \times [c, d]$ is defined to be

$$\iint_{[a, b] \times [c, d]} f(x, y) dA = \lim_{\substack{\max \Delta x_i \rightarrow 0 \\ \max \Delta y_j \rightarrow 0}} \sum_{i=1}^m \sum_{j=1}^n f(u_i, v_j) \Delta x_i \Delta y_j,$$

where (u_i, v_j) is some point in the rectangle

$(x_{i-1}, x_i) \times (y_{j-1}, y_j)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.



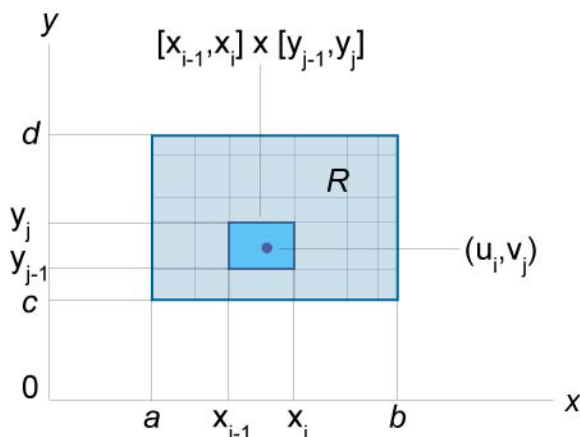


Figure 189.

The double integral over a general region R is

$$\iint_R f(x, y) dA = \iint_{[a, b] \times [c, d]} g(x, y) dA,$$

where rectangle $[a, b] \times [c, d]$ contains R ,

$g(x, y) = f(x, y)$ if $f(x, y)$ is in R and $g(x, y) = 0$ otherwise.

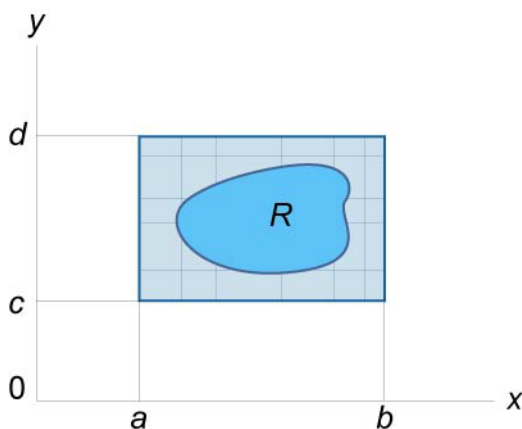


Figure 190.

$$1079. \iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$$

$$1080. \iint_R [f(x, y) - g(x, y)] dA = \iint_R f(x, y) dA - \iint_R g(x, y) dA$$

$$1081. \iint_R kf(x, y) dA = k \iint_R f(x, y) dA,$$

where k is a constant.

$$1082. \text{ If } f(x, y) \leq g(x, y) \text{ on } R, \text{ then } \iint_R f(x, y) dA \leq \iint_R g(x, y) dA.$$

1083. If $f(x, y) \geq 0$ on R and $S \subset R$, then

$$\iint_S f(x, y) dA \leq \iint_R f(x, y) dA.$$

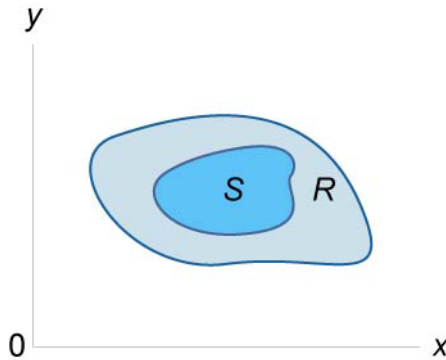


Figure 191.

$$1084. \text{ If } f(x, y) \geq 0 \text{ on } R \text{ and } R \text{ and } S \text{ are non-overlapping regions, then } \iint_{R \cup S} f(x, y) dA = \iint_R f(x, y) dA + \iint_S f(x, y) dA.$$

Here $R \cup S$ is the union of the regions R and S .

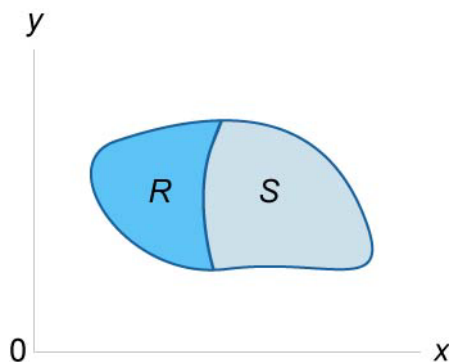


Figure 192.

1085. Iterated Integrals and Fubini's Theorem

$$\iint_R f(x, y) dA = \int_a^b \int_{p(x)}^{q(x)} f(x, y) dy dx$$

for a region of type I,

$$R = \{(x, y) | a \leq x \leq b, p(x) \leq y \leq q(x)\}.$$

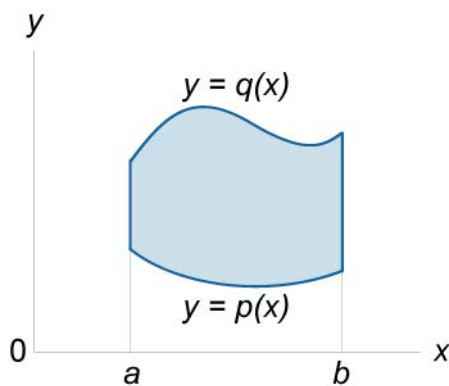


Figure 193.

$$\iint_R f(x, y) dA = \int_c^d \int_{u(y)}^{v(y)} f(x, y) dx dy$$

for a region of type II,

$$R = \{(x, y) | u(y) \leq x \leq v(y), c \leq y \leq d\}.$$

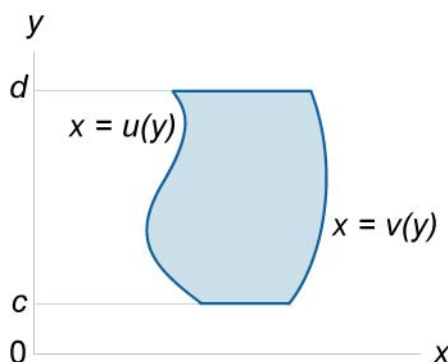


Figure 194.

1086. Double Integrals over Rectangular Regions

If R is the rectangular region $[a, b] \times [c, d]$, then

$$\iint_R f(x, y) dx dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$$

In the special case where the integrand $f(x, y)$ can be written as $g(x)h(y)$ we have

$$\iint_R f(x, y) dx dy = \iint_R g(x)h(y) dx dy = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right).$$

1087. Change of Variables

$$\iint_R f(x, y) dx dy = \iint_S f[x(u, v), y(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$ is the **jacobian** of the trans-

formations $(x, y) \rightarrow (u, v)$, and S is the pullback of R which

can be computed by $x = x(u, v)$, $y = y(u, v)$ into the definition of R .

1088. Polar Coordinates

$$x = r \cos \theta, \quad y = r \sin \theta.$$

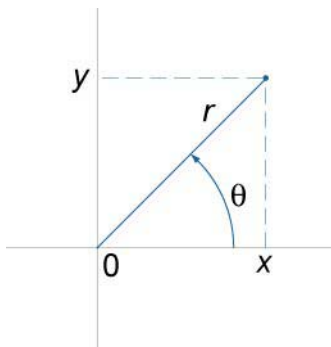


Figure 195.

1089. Double Integrals in Polar Coordinates

The Differential $dx dy$ for Polar Coordinates is

$$dx dy = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = r dr d\theta.$$

Let the region R is determined as follows:

$$0 \leq g(\theta) \leq r \leq h(\theta), \quad \alpha \leq \theta \leq \beta, \quad \text{where } \beta - \alpha \leq 2\pi.$$

Then

$$\iint_R f(x, y) dx dy = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

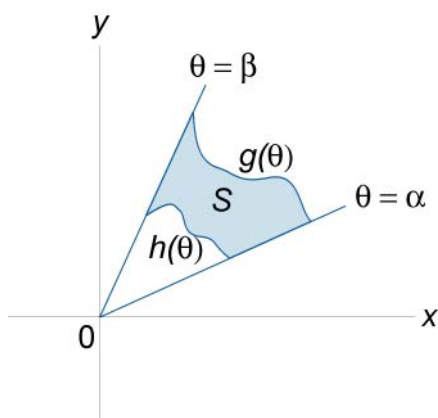


Figure 196.

If the region R is the **polar rectangle** given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $\beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) dx dy = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta.$$

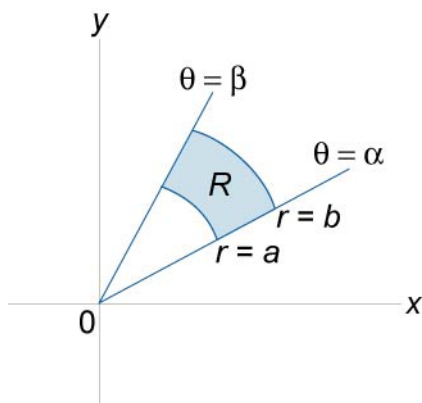


Figure 197.

1090. Area of a Region

$$A = \int_a^b \int_{g(x)}^{f(x)} dy dx \quad (\text{for a type I region}).$$

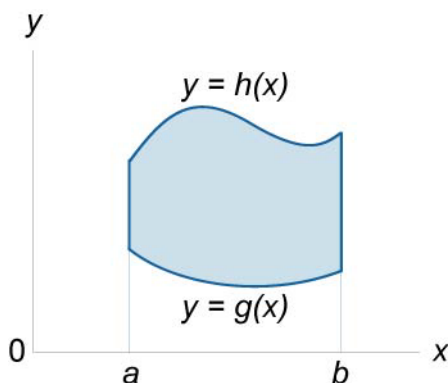


Figure 198.

$$A = \int_c^d \int_{p(y)}^{q(y)} dx dy \quad (\text{for a type II region}).$$

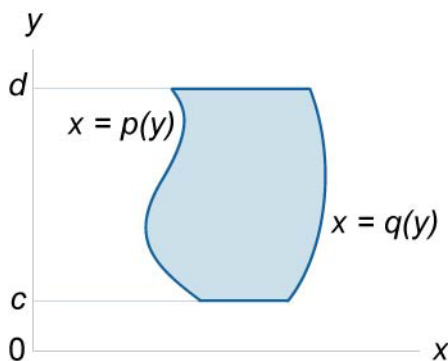


Figure 199.

1091. Volume of a Solid

$$V = \iint_R f(x, y) dA.$$

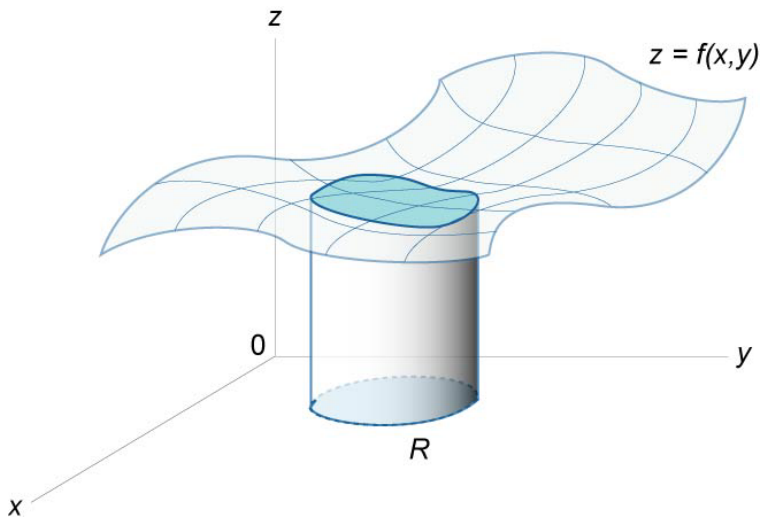


Figure 200.

If R is a type I region bounded by $x = a$, $x = b$, $y = h(x)$, $y = g(x)$, then

$$V = \iint_R f(x, y) dA = \int_a^b \int_{h(x)}^{g(x)} f(x, y) dy dx.$$

If R is a type II region bounded by $y = c$, $y = d$, $x = q(y)$, $x = p(y)$, then

$$V = \iint_R f(x, y) dA = \int_c^d \int_{p(y)}^{q(y)} f(x, y) dx dy.$$

If $f(x,y) \geq g(x,y)$ over a region R , then the volume of the solid between $z_1 = f(x,y)$ and $z_2 = g(x,y)$ over R is given by

$$V = \iint_R [f(x,y) - g(x,y)] dA.$$

1092. Area and Volume in Polar Coordinates

If S is a region in the xy -plane bounded by $\theta = \alpha$, $\theta = \beta$, $r = h(\theta)$, $r = g(\theta)$, then

$$A = \iint_S dA = \int_{\alpha}^{\beta} \int_{h(\theta)}^{g(\theta)} r dr d\theta,$$

$$V = \iint_S f(r,\theta) r dr d\theta.$$

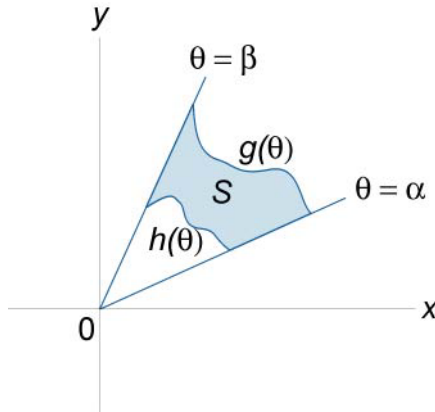


Figure 201.

1093. Surface Area

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$

1094. Mass of a Lamina

$$m = \iint_R \rho(x, y) dA,$$

where the lamina occupies a region R and its density at a point (x, y) is $\rho(x, y)$.

1095. Moments

The moment of the lamina about the x -axis is given by formula

$$M_x = \iint_R y \rho(x, y) dA.$$

The moment of the lamina about the y -axis is

$$M_y = \iint_R x \rho(x, y) dA.$$

The moment of inertia about the x -axis is

$$I_x = \iint_R y^2 \rho(x, y) dA.$$

The moment of inertia about the y -axis is

$$I_y = \iint_R x^2 \rho(x, y) dA.$$

The polar moment of inertia is

$$I_0 = \iint_R (x^2 + y^2) \rho(x, y) dA.$$

1096. Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R x \rho(x, y) dA = \frac{\iint_R x \rho(x, y) dA}{\iint_R \rho(x, y) dA},$$



$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R y \rho(x, y) dA = \frac{\iint_R y \rho(x, y) dA}{\iint_R \rho(x, y) dA}.$$

1097. Charge of a Plate

$$Q = \iint_R \sigma(x, y) dA,$$

where electrical charge is distributed over a region R and its charge density at a point (x, y) is $\sigma(x, y)$.

1098. Average of a Function

$$\mu = \frac{1}{S} \iint_R f(x, y) dA,$$

$$\text{where } S = \iint_R dA.$$

